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SPACE—TIME—MATTER

BY

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we get that *the energy, J_0, or the inertial mass of the system, is equal to the mass m_0, which is characteristic of the gravitational field generated by the system* (*vide* note 28). On the other hand it is to be remarked parenthetically that the physics based on the notion of substance leads to the space-integral of μ/f for the value of the mass, whereas, in reality, for incoherent matter $J_0 = m_0 =$ the space-integral of μ; this is a definite indication of how radically erroneous is the whole idea of substance.

§ 34. Concerning the Inter-connection of the World as a Whole

The general theory of relativity leaves it quite undecided whether the world-points may be represented by the values of four co-ordinates x_i in a singly reversible continuous manner or not. It merely assumes that the **neighbourhood** of every world-point admits of a singly reversible continuous representation in a region of the four-dimensional "number-space" (whereby "point of the four-dimensional number-space" is to signify any number-quadruple); it makes no assumptions at the outset about the inter-connection of the world. When, in the theory of surfaces, we start with a parametric representation of the surface to be investigated, we are referring only to a piece of the surface, not to the whole surface, which in general can by no means be represented uniquely and continuously on the Euclidean plane or by a plane region. Those properties of surfaces that persist during all one-to-one continuous transformations form the subject-matter of *analysis situs* (the analysis of position); connectivity, for example, is a property of analysis situs. Every surface that is generated from the sphere by continuous deformation does not, from the point of view of analysis situs, differ from the sphere, but does differ from an anchor-ring, for instance. For on the anchor-ring there exist closed lines, which do not divide it into several regions, whereas such lines are not to be found on the sphere. From the geometry which is valid on a sphere, we derived "spherical geometry" (which, following Riemann, we set up in contrast with the geometry of Bolyai-Lobatschefsky) by identifying two diametrically opposite points of the sphere. The resulting surface **F** is from the point of view of *analysis situs* likewise different from the sphere, in virtue of which property it is called one-sided. If we imagine on a surface a small wheel in continual rotation in the one direction to be moved along this surface during the rotation, the centre of the wheel describing a closed curve, then we should expect that when the wheel has returned to its initial position it would rotate in the same direction as at the commencement of its motion. If this is the case, then whatever curve the centre of the wheel may have described on the surface, the latter is called **two-sided**; in the reverse case, it is called **one-sided.** The existence of one-sided surfaces was first pointed out by Möbius. The surface **F** mentioned above is two-sided, whereas the sphere is, of course, one-sided. This is obvious if the centre of the wheel be made to describe a great circle; on the sphere the **whole** circle must be traversed if this path is to be closed, whereas on **F** only the half need be covered. Quite analogously to the case of two-dimensional manifolds, four-dimensional ones may be endowed with diverse properties with regard to *analysis situs*. But in every four-dimensional manifold the neighbourhood of a point may, of course, be represented in a continuous manner by four co-ordinates in such a way that different co-ordinate quadruples always correspond to different points of this neighbourhood. The use of the four world-co-ordinates is to be interpreted in just this way.

Every world-point is the origin of the double-cone of the active future and the passive past. Whereas in the special theory of relativity these two portions are separated by an intervening region, it is certainly possible in the present case for the cone of the active future to overlap with that of the passive past; so that, in principle, it is possible to experience events now that will in part be an effect of my future resolves and actions. Moreover, it is not impossible for a world-line (in particular, that of my body), although it has a time-like direction at every point, to return to the neighbourhood of a point which it has already once passed through. The result would be a spectral image of the world more fearful than anything the weird fantasy of E. T. A. Hoffmann has ever conjured up. In actual fact the very considerable fluctuations of the g_{ik}'s that would be necessary to produce this effect do not occur in the region of world in which we live. Nevertheless there is a certain amount of interest in speculating on these possibilities inasmuch as they shed light on the philosophical problem of cosmic and phenomenal time. Although paradoxes of this kind appear, nowhere do we find any real contradiction to the facts directly presented to us in experience.

We saw in § 26 that, apart from the consideration of gravitation, the fundamental electrodynamic laws (of Mie) have a form such as is demanded by the **principle of causality.** The time-derivatives of the phase-quantities are expressed in terms of these quantities themselves and their spatial differential co-efficients. These facts persist when we introduce gravitation and thereby increase the table of phase-quantities ϕ_i, F_{ik}, by the g_{ik}'s and the

$\left\{ {ik \atop r} \right\}$'s. But on account of the general invariance of physical laws we must formulate our statements so that, from the values of the phase-quantities for one moment, all those assertions concerning them, **which have an invariant character,** follow as a consequence of physical laws; moreover, it must be noted that this statement does not refer to the world as a whole but only to a portion which can be represented by four co-ordinates. Following Hilbert (*vide* note 29) we proceed thus. In the neighbourhood of the world-point O we introduce 4 co-ordinates x_i, such that, at O itself,

$$ds^2 = dx_0^2 - (dx_1^2 + dx_2^2 + dx_3^2).$$

In the three-dimensional space $x_0 = 0$ surrounding O we may mark off a region **R**, such that, in it, $-ds^2$ remains definitely positive. Through every point of this region we draw the geodetic world-line which is orthogonal to that region, and which has a time-like direction. These lines will cover singly a certain four-dimensional neighbourhood of O. We now introduce new co-ordinates which will coincide with the previous ones in the three-dimensional space **R**, for we shall now assign the co-ordinates x_0, x_1, x_2, x_3 to the point P at which we arrive, if we go from the point $P_0 = (x_1, x_2, x_3)$ in **R** along the orthogonal geodetic line passing through it, so far that the proper-time of the arc traversed, P_0P, is equal to x_0. This system of co-ordinates was introduced into the theory of surfaces by Gauss. Since $ds^2 = dx_0^2$ on each of the geodetic lines, we must get identically for all four co-ordinates in this co-ordinate system:

$$g_{00} = 1 \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Since the lines are orthogonal to the three-dimensional space $x_0 = 0$, we get for $x_0 = 0$

$$g_{01} = g_{02} = g_{03} = 0 \quad . \quad . \quad . \quad . \quad (59)$$

Moreover, since the lines that are obtained when x_1, x_2, x_3 are kept constant and x_0 is varied are geodetic, it follows (from the equation of geodetic lines) that

$$\left\{ {00 \atop i} \right\} = 0 \qquad (i = 0, 1, 2, 3)$$

and hence also that

$$\left[{00 \atop i} \right] = 0.$$

Taking (58) into consideration, we get from the latter

$$\frac{\partial g_0}{\partial x_0} = 0 \qquad (i = 1, 2, 3)$$

and, on account of (59), we have consequently not only for $x_0 = 0$ but also identically for the four co-ordinates that

$$g_{0i} = 0 \qquad (i = 1, 2, 3) \; . \quad (60)$$

The following picture presents itself to us: a family of geodetic lines with time-like direction which covers a certain world-region singly and completely (without gaps); also, a similar uni-parametric family of three-dimensional spaces $x_0 = \text{const.}$ According to (60) these two families are everywhere orthogonal to one another, and all portions of arc cut off from the geodetic lines by two of the "parallel" spaces $x_0 = \text{const.}$ have the same proper-time. If we use this particular co-ordinate system, then

$$\frac{\partial g_{ik}}{\partial x_0} = -2 \begin{Bmatrix} ik \\ 0 \end{Bmatrix} \quad (i, k = 1, 2, 3)$$

and the gravitational equations enable us to express the derivatives

$$\frac{\partial}{\partial x_0} \begin{Bmatrix} ik \\ 0 \end{Bmatrix} \qquad (i, k = 1, 2, 3)$$

not only in terms of the ϕ_i's and their derivatives, but also in terms of the g_{ik}'s, their derivatives (of the first and second order) with respect to x_1, x_2, x_3, and the $\begin{Bmatrix} ik \\ 0 \end{Bmatrix}$'s themselves.

Hence, by regarding the twelve quantities,

$$g_{ik}, \qquad \begin{Bmatrix} ik \\ 0 \end{Bmatrix} \qquad (i, k = 1, 2, 3)$$

together with the electromagnetic quantities, as the unknowns, we arrive at the required result (x_0 playing the part of time). The cone of the passive past starting from the point O' with a positive x_0 co-ordinate will cut a certain portion $\mathbf{R}'$ out of $\mathbf{R}$, which, with the sheet of the cone, will mark off a finite region of the world $\mathbf{G}$ (namely, a conical cap with its vertex at O'). If our assertion that the geodetic null-lines denote the initial points of all action is rigorously true, then the values of the above twelve quantities as well as the electromagnetic potentials ϕ_i and the field-quantities F_{ik} in the three-dimensional region of space $\mathbf{R}'$ determine fully the values of the two latter quantities in the world-region $\mathbf{G}$. This has hitherto not been proved. *In any case, we see that the differential equations of the field contain the physical laws of nature in their complete form*, and that there cannot be a further limitation due to boundary conditions at spatial infinity, for example.

Einstein, arguing from cosmological considerations of the inter-connection of the world as a whole (*vide* note 30) came to the conclusion that the world is finite in space. Just as in the Newtonian theory of gravitation the law of contiguous action expressed in Poisson's equation entails the Newtonian law of attraction only if the condition that the gravitational potential vanishes at infinity is superimposed, so Einstein in his theory seeks to supplement the differential equations by introducing boundary conditions at spatial infinity. To overcome the difficulty of formulating conditions of a general invariant character, which are in agreement with astronomical facts, he finds himself constrained to assume that the world is closed with respect to space; for in this case the boundary conditions are absent. In consequence of the above remarks the author cannot admit the cogency of this deduction, since the differential equations in themselves, without boundary conditions, contain the physical laws of nature in an unabbreviated form excluding every ambiguity. So much more weight is accordingly to be attached to another consideration which arises from the question: How does it come about that our stellar system with the relative velocities of the stars, which are extraordinarily small in comparison with that of light, persists and maintains itself and has not, even ages ago, dispersed itself into infinite space? This system presents exactly the same view as that which a molecule in a gas in equilibrium offers to an observer of correspondingly small dimensions. In a gas, too, the individual molecules are not at rest but the small velocities, according to Maxwell's law of distribution, occur much more often than the large ones, and the distribution of the molecules over the volume of the gas is, on the average, uniform, so that perceptible differences of density occur very seldom. If this analogy is legitimate, we could interpret the state of the stellar system and its gravitational field according to the same **statistical principles** that tell us that an isolated volume of gas is almost always in equilibrium. This would, however, be possible only if the **uniform distribution of stars at rest in a static gravitational field, as an ideal state of equilibrium,** is reconcilable with the laws of gravitation. In a statical field of gravitation the world-line of a point-mass at rest, that is, a line on which x_1, x_2, x_3 remain constant and x_0 alone varies, is a geodetic line if

$$\left\{ {00 \atop i} \right\} = 0, \qquad (i = 1, 2, 3)$$

and hence

$$\begin{bmatrix} 00 \\ i \end{bmatrix} = 0 \qquad \frac{\partial g_{00}}{dx_i} = 0.$$

Therefore, a distribution of mass at rest is possible only if

$$\sqrt{g_{00}} = f = \text{const.} = 1.$$

The equation

$$\Delta f = \tfrac{1}{2}\mu \qquad (\mu = \text{density of mass}) \quad . \quad . \quad . \quad (32)$$

then shows, however, that the ideal state of equilibrium under consideration **is incompatible** with the laws of gravitation, as hitherto assumed.

In deriving the gravitational equations in § 28, however, we committed a sin of omission. R is not the only invariant dependent on the g_{ik}'s and their first and second differential co-efficients, and which is linear in the latter; for the most general invariant of this description has the form $aR + \beta$, in which a and β are numerical constants. Consequently we may generalise the laws of gravitation by replacing R by $R + \lambda$ (and $\mathbf{G}$ by $\mathbf{G} + \frac{1}{2}\lambda\sqrt{g}$), in which λ denotes a universal constant. If it is not equal to 0, as we have hitherto assumed, we may take it equal to 1; by this means not only has the unit of time been reduced by the principle of relativity, to the unit of length, and the unit of mass by the law of gravitation to the same unit, but the unit of length itself is fixed absolutely. With these modifications the gravitational equations for statical non-coherent matter ($\mathbf{T}_0^0 = \mu = \mu_0\sqrt{g}$, all other components of the tensor-density $\mathbf{T}$ being equal to zero) give, if we use the equation $f = 1$ and the notation of § 29:

$$\lambda = \mu_0 \text{ [in place of (32)]}$$

and

$$P_{ik} - \lambda\gamma_{ik} = 0 \qquad (i, k = 1, 2, 3) \quad . \quad . \quad (61)$$

Hence this ideal state of equilibrium is possible under these circumstances if the mass is distributed with the density λ. The space must then be homogeneous metrically; and indeed the equations (61) are then actually satisfied for a spherical space of radius $a = \sqrt{2/\lambda}$. Thus, in space, we may introduce four co-ordinates, connected by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2, \quad . \quad . \quad . \quad (62)$$

for which we get

$$d\sigma^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2.$$

From this we conclude that space is closed and hence finite. If this were not the case, it would scarcely be possible to imagine how a state of statistical equilibrium could come about. If the world is closed, spatially, it becomes possible for an observer to see several pictures of one and the same star. These depict the star at epochs separated by enormous intervals of time (during which light travels once entirely round the world). We have yet to inquire whether the points of space correspond singly and reversibly to the value-quadruples x_i which satisfy the condition (62), or whether two value-systems

$$(x_1, x_2, x_3, x_4) \text{ and } (-x_1, -x_2, -x_3, -x_4)$$

correspond to the same point. From the point of view of *analysis situs* these two possibilities are different even if both spaces are two-sided. According as the one or the other holds, the total mass of the world in grammes would be

$$\frac{\pi a}{2\kappa} \text{ or } \frac{\pi a}{4\kappa}, \text{ respectively.}$$

Thus our interpretation demands that the total mass that happens to be present in the world bear a definite relation to the universal constant $\lambda = \frac{2}{a^2}$ which occurs in the law of action; this obviously makes great demands on our credulity.

The radially symmetrical solutions of the modified homogeneous equations of gravitation that would correspond to a world empty of mass are derivable by means of the principle of variation (*vide* § 31 for the notation)

$$\delta\int(2w\Delta' + \lambda\Delta r^2)dr = 0.$$

The variation of w gives, as earlier, $\Delta = 1$. On the other hand, variation of Δ gives

$$w' = \frac{\lambda}{2}r^2 \quad . \quad . \quad . \quad . \quad . \quad (63)$$

If we demand regularity at $r = 0$, it follows from (63) that

$$w = \frac{\lambda}{6}r^3$$

$$\text{and } \frac{1}{h^2} = f^2 = 1 - \frac{\lambda}{6}r^2 \quad . \quad . \quad . \quad (64)$$

The space may be represented congruently on a "sphere"

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 3a^2 \quad . \quad . \quad . \quad (65)$$

of radius $a\sqrt{3}$ in four-dimensional Euclidean space (whereby one of the two poles on the sphere, whose first three co-ordinates, x_1, x_2, x_3 each $= 0$, corresponds to the centre in our case). The world is a cylinder erected on this sphere in the direction of a fifth co-ordinate axis t. But since on the "greatest sphere" $x_4 = 0$, which may be designated as the equator or the space-horizon for that centre, f becomes zero, and hence the metrical groundform of the world becomes singular, we see that the possibility of a stationary empty world is contrary to the physical laws that are here regarded as valid. There must at least be masses at the horizon. The calculation may be performed most readily if (merely to orient ourselves on the question) we assume an incompressible fluid to be present there. According to § 32 the problem of variation that is to be solved is (if we use the same notation and add the λ term)

$$\delta\int\left\{\Delta' w + \left(\mu_0 + \frac{\lambda}{2}\right)r^2\Delta - r^2 vh\right\}dr = 0.$$

In comparison with the earlier expression we note that the only change consists in the constant μ_0 being replaced by $\mu_0 + \frac{\lambda}{2}$. As earlier, it follows that

$$w' - \left(\mu_0 + \frac{\lambda}{2}\right)r^2 = 0,\ w = -2M + \frac{2\mu_0 + \lambda}{6}r^3,$$

$$\frac{1}{h^2} = 1 + \frac{2M}{r} - \frac{2\mu_0 + \lambda}{6}r^2 \quad . \quad . \quad . \quad (66)$$

If the fluid is situated between the two meridians $x_4 =$ const., which have a radius r_0 ($< a\sqrt{3}$), then continuity of argument with (64) demands that the constant

$$M = \frac{\mu_0}{6}r_0^3.$$

To the first order $\frac{1}{h^2}$ becomes equal to zero for a value $r = b$ between r_0 and $a\sqrt{3}$. Hence the space may still be represented on the sphere (65), but this representation is no longer congruent for the zone occupied by fluid. The equation for Δ (p. 265) now yields a value of f that does not vanish at the equator. The boundary condition of vanishing pressure gives a transcendental relation between μ_0 and r_0, from which it follows that, if the mass-horizon is to be taken arbitrarily small, then the fluid that comes into question must have a correspondingly great density, namely, such that the total mass does not become less than a certain positive limit (*vide* note 31).

The general solution of (63) is

$$\frac{1}{h^2} = f^2 = 1 - \frac{2m}{r} - \frac{\lambda}{6}r^2 \quad (m = \text{const.}).$$

It corresponds to the case in which a spherical mass is situated at the centre. The world can be empty of mass only in a zone $r_0 \leqq r \leqq r_1$, in which this f^2 is positive; a mass-horizon is again necessary. Similarly, if the central mass is charged electrically; for in this case, too, $\Delta = 1$. In the expression for $\frac{1}{h^2} = f^2$ the electrical term $+\frac{e^2}{r^2}$ has to be added, and the electrostatic potential $=\frac{e}{r}$.

Perhaps in pursuing the above reflections we have yielded too readily to the allurement of an imaginary flight into the region of masslessness. Yet these considerations help to make clear what the new views of space and time bring within the realm of **possibility.** The assumption on which they are based is at any rate the simplest on which it becomes explicable that, in the world as actually presented to us, statical conditions obtain as a whole, so far as the electromagnetic and the gravitational field is concerned, and that just those solutions of the statical equations are valid which vanish at infinity or, respectively, converge towards Euclidean metrics. For on the sphere these equations will have a unique solution (boundary conditions do not enter into the question as they are replaced by the postulate of regularity over the whole of the closed configuration). If we make the constant λ arbitrarily small, the spherical solution converges to that which satisfies at infinity the boundary conditions mentioned for the infinite world which results when we pass to the limit.

A metrically homogeneous world is obtained most simply if, in a five-dimensional space with the metrical groundform $ds^2 = -\Omega(dx)$, ($-\Omega$ denotes a non-degenerate quadratic form with constant co-efficients), we examine the four-dimensional "conic-section" defined by the equation $\Omega(x) = \frac{6}{\lambda}$. Thus this basis gives us a solution of the Einstein equations of gravitation, modified by the λ term, for the case of no mass. If, as must be the case, the resulting metrical groundform of the world is to have one positive and three negative dimensions, we must take for Ω a form with four positive dimensions and one negative, thus

$$\Omega(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2.$$

By means of a simple substitution this solution may easily be transformed into the one found above for the statical case. For if we set

$$x_4 = z \cosh t, \qquad x_5 = z \sinh t$$

we get

$$x_1^2 + x_2^2 + x_3^2 + z^2 = \frac{6}{\lambda}, \quad -ds^2 = (dx_1^2 + dx_2^2 + dx_3^2 + dz^2) - z^2 dt^2.$$

These "new" z, t co-ordinates, however, enable only the "wedge-shaped" section $x_4^2 - x_5^2 > 0$ to be represented. At the "edge" of the wedge (at which $x_4 = 0$ simultaneously with $x_5 = 0$), t becomes indeterminate. This edge, which appears as a two-dimensional configuration in the original co-ordinates is, therefore, three-dimensional in the new co-ordinates; it is the cylinder erected in the direction of the t-axis over the equator $z = 0$ of the sphere (65). The question arises whether it is the first or the second co-ordinate system that serves to represent the whole world in a regular manner. In the former case the world would not be static as a whole, and the absence of matter in it would be in agreement with physical laws; de Sitter argues from this assumption (*vide* note 32). In the latter case we have a static world that cannot exist without a mass-horizon; this assumption, which we have treated more fully, is favoured by Einstein.

§ 35. The Metrical Structure of the World as the Origin of Electromagnetic Phenomena *

We now aim at a final synthesis. To be able to characterise the physical state of the world at a certain point of it by means of numbers we must not only refer the neighbourhood of this point to a co-ordinate system but we must also fix on certain units of measure. We wish to achieve just as fundamental a point of view with regard to this second circumstance as is secured for the first one, namely, the arbitrariness of the co-ordinate system, by the Einstein Theory that was described in the preceding paragraph. This idea, when applied to geometry and the conception of distance (in Chapter II) after the step from Euclidean to Riemann geometry had been taken, effected the final entrance into the realm of infinitesimal geometry. Removing every vestige of ideas of "action at a distance," let us assume that world-geometry is of this kind; we then find that the metrical structure of the world, besides being dependent on the quadratic form (1), is also dependent on a linear differential form $\phi_i \, dx_i$.

Just as the step which led from the special to the general theory of relativity, so this extension affects immediately only the world-geometrical foundation of physics. Newtonian mechanics, as also the special theory of relativity, assumed that uniform translation is a unique state of motion of a set of vector axes, and hence that the position of the axes at one moment determines their position in all other moments. But this is incompatible with the intuitive principle of the **relativity of motion.** This principle could be satisfied, if facts are not to be violated drastically, only by maintaining the conception of **infinitesimal** parallel displacement of a vector set of axes; but we found ourselves obliged to regard the

* *Vide* note 33.